Performance analysis in wireless communications and large deviations of extreme eigenvalues of deformed random matrices

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  - Presentation of the models
  - General results
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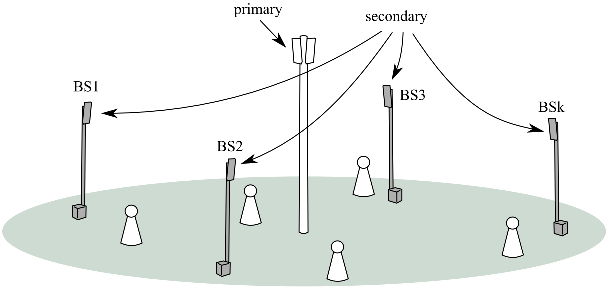
  - General results
  - Application to some classical models
- Conclusion

Source detection in cooperative spectrum sensing



Secondary sensors try to find a bandwidth to occupy. Those **K** sensors can share information, each of them receiving **N** samples of the signal.

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- **Hypothesis H1 : Presence of a signal.** The data received by sensor number k is now of the form :

$$y_k(n) = h_k \, s(n) + w_k(n) \; , \quad n = 1 \dots N$$

where $s(n)$ is a Gaussian primary signal and h_k the fading coefficient associated to the secondary sensor k.

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As σ and $\mathbf{h}$ are unknown, the Neyman-Pearson test cannot be implemented.

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$$p_0(\mathbf{Y}; \sigma^2) = (\pi\sigma^2)^{-NK} \exp\left(-\frac{N}{\sigma^2} \mathrm{tr}\, \mathbf{R}\right) .$$

  where $\mathbf{R} = \frac{1}{N}\mathbf{Y}\mathbf{Y}^*$ is the empirical covariance matrix.

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- Under **H1**, the column vectors of $\mathbf{Y}$ are i.i.d. $\mathcal{CN}(0, \mathbf{h}\mathbf{h}^* + \sigma^2 \mathbf{I}_K)$ where $\mathbf{h} = [h_1, \ldots, h_K]^T$ is the fading vector corresponding to the K secondary sensors. The likelihood writes :

  $$p_1(\mathbf{Y}; \mathbf{h}, \sigma^2) = (\pi^K \det(\mathbf{h}\mathbf{h}^* + \sigma^2 \mathbf{I}_K))^{-N} \exp\left(-N \mathrm{tr}\left(\mathbf{R}(\mathbf{h}\mathbf{h}^* + \sigma^2 \mathbf{I}_K)^{-1}\right)\right) \ .$$

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Recall that $\sigma^2, \mathbf{h}$ are unknown. The GLRT will reject **H0** for high values of the statistics :

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After some standard computations, we get the following test :

Reject **H0** whenever the statistics :

$$T_N := \frac{\lambda_{\max}}{\frac{1}{K}\mathrm{tr}\mathbf{R}}$$

is **above** the threshold γ

where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{R} := \frac{1}{N}\mathbf{Y}\mathbf{Y}^*$.

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For a given threshold γ, we define :

- the type I Error (probability of false alarm) $P_0[T_N > \gamma]$ is the probability of deciding **H1** when **H0** holds,
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The Receiver Operating Characterictic (ROC curve) is the set of points (Type I Error, Type II Error) for all possible thresholds.

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$\Rightarrow$ We study the ROC curve in the asymptotic regime :

$$K \to \infty, \ N \to \infty, \frac{K}{N} \to c \ \in (0, 1)$$

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$\Rightarrow$ This determines the asymptotic threshold γ for a fixed PFA.

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- Consequently, $T_N = \frac{\lambda_{\max}}{\frac{1}{K}\, tr\, \mathbf{R}}$ converges to

$$\lambda_{\text{spiked}} := (1+\rho)\left(1+\frac{c}{\rho}\right) > (1+\sqrt{c})^2 := \lambda^+ .$$

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Otherwise stated,

$$\begin{aligned} P_0[T_N > \gamma] &\simeq e^{-N\,\mathcal{E}_0(\gamma)} \\ P_1[T_N < \gamma] &\simeq e^{-N\,\mathcal{E}_1(\gamma)} \; . \end{aligned}$$

The set of couples $(\mathcal{E}_0(\gamma), \mathcal{E}_1(\gamma))$ is called **asymptotic error exponent curve**

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  Deviations of $\lambda_{\max}$ under H_1 ("spiked" model) (cf Maïda)

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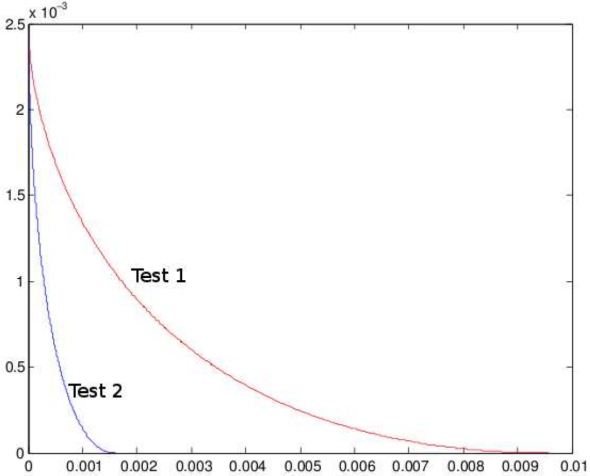

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and U_i^n obtained by orthonormalization.

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This means that for any open set $O \subset \mathbb{R}^{r_0}$,

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Important generalisation : we can relax the hypothesis on the extreme eigenvalues, provided the law of $\frac{G}{\sqrt{n}}$ satisfies a LDP.

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Consider the i.i.d. case when $X_n = 0$. If G_n are $n \times r$ matrices whose rows are i.i.d. copies of G and $\Theta = diag(\theta_1, \ldots, \theta_r)$, we can study the eigenvalues of $W_n = \frac{1}{n} G_n^* \Theta G_n$ (see Fey, van der Hofstad, Klok, $\Theta = Id$).

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In the standard case,

$$L(\alpha_1, \ldots, \alpha_r) = \frac{1}{2} \sum_{i=1}^{r} \left(\frac{\alpha_i}{\theta_i} - 1 - \log \frac{\alpha_i}{\theta_i} \right).$$

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From there it is easy to deduce the rate function for the largest eigenvalue

$$L_{max}(x) = \begin{cases} \frac{1}{2}(x - 1 - \log x) & \text{if } x \geqslant 1 \\ \frac{r}{2}(x - 1 - \log x) & \text{if } x \in (0,1) \end{cases}$$

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Under appropriate assumptions on V, for any fixed k, the law of the k largest eigenvalues of $\widetilde{X_n}$ satisfies a large deviation principle with a good rate function.

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Theorem
Under appropriate assumptions on V, for any fixed k, the law of the k largest eigenvalues of $\widetilde{X_n}$ satisfies a large deviation principle with a good rate function.

Rq : we first condition on the deviation of the eigenvalues of X_n so that we can consider those as outliers.

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$$f_n(z) = det\left(\left[G_{i,j}^n(z)\right]_{i,j=1}^r - diag\left(\theta_1^{-1}, \ldots, \theta_r^{-1}\right)\right),$$

with

$$G_{i,j}^n(z) = \langle U_i^n, (z - X_n)^{-1} U_j^n \rangle.$$

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$$f_n(z) = det\left(\left[G^n_{i,j}(z)\right]^r_{i,j=1} - diag\left(\theta_1^{-1}, \ldots, \theta_r^{-1}\right)\right),$$

with

$$G^n_{i,j}(z) = \langle U^n_i, (z - X_n)^{-1} U^n_j \rangle.$$

If

$$K^n_{i,j}(z) = \langle G^n_i, (z - X_n)^{-1} G^n_j \rangle = \frac{1}{n}\sum_{k=1}^{n} \frac{g_i(k)\overline{g_j(k)}}{z - \lambda_k}$$

and $C^n_{i,j} = \frac{1}{n}\sum_{k=1}^{n} g_i(k)\overline{g_j(k)}$ then $f_n(z) = P_{\Theta,r}(K^n(z), C^n)$

Conclusion

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Can we use large deviation principles of this type to analyse the performance for some other models relevant in wireless communication context ?