Norm of polynomials in Large Random Matrices

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Introduction

The Gaussian Unitary Ensemble (GUE)

Definition

We said that $X^{(N)}$ is an $N \times N$ GUE matrix if $X^{(N)} = X^{(N)*}$ with entries $X^{(N)} = (X_{n,m})_{1 \leqslant n,m \leqslant N}$, where

$$\Big((X_{n,n})_{1 \leqslant n \leqslant N}, (\sqrt{2}\mathrm{Re}\ (X_{n,m}), \sqrt{2}\mathrm{Im}\ (X_{n,m})\ )_{1 \leqslant n < m \leqslant N} \Big)$$

is a centered Gaussian vector with covariance matrix $\frac{1}{N}\mathbf{1}_{N^2}$.

Classical results

Let $X_N \sim$ GUE. Denote the eigenvalues of $X^{(N)}$ by $\lambda_1 \leqslant \ldots \leqslant \lambda_N$.

Theorem (Wigner 55)

The empirical spectral measure of $X^{(N)}$

$$L(X^{(N)}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

converges when $N \to \infty$ to the semicircular law with radius 2.

Theorem

When $N \to \infty$

$$\lambda_1 \to -2, \quad \lambda_N \to 2.$$

Reformulation

- Convergence of $L(X^{(N)})$: a.s. and in $\mathbb{E}$ in moments

$$L_N(P) = \frac{1}{N}\sum_{i=1}^N P(\lambda_i) = \frac{1}{N}\mathrm{Tr}\big[P(X^{(N)})\ \big] \underset{N\to\infty}{\longrightarrow} \tau[P] := \int P d\sigma,$$

for all polynomial P, with $d\sigma(t) = \frac{1}{2\pi}\sqrt{4-t^2}\ \mathbf{1}_{|t|\leqslant 2}\ dt$ the semicircle distribution.

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for all polynomial P, with $d\sigma(t) = \frac{1}{2\pi}\sqrt{4-t^2}\ \mathbf{1}_{|t|\leqslant 2}\ dt$ the semicircle distribution.

- Convergence of extremal eigenvalues : a.s.

$$\|X^{(N)}\| \underset{N\to\infty}{\longrightarrow} 2,$$

with $\|\cdot\|$ the operator norm:

$$\begin{aligned} \|M\| &= \sqrt{\rho(M^*M)} \\ &= \rho(M) \text{ if } M \text{ Hermitian} \end{aligned}$$

where ρ is the spectral radius.

The context of this talk

The protagonists

- $\mathbf{X}_N = (X_1^{(N)}, \dots, X_p^{(N)})$ family of independent $N \times N$ GUE matrices,
- $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_q^{(N)})$ family of arbitrary $N \times N$ matrices.

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We want to

- extend such results for matrices of the form

$$M_N = P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*),$$

where P is any non commutative polynomial in $p + 2q$ indeterminates,

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- extend such results for matrices of the form

$$M_N = P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*),$$

  where P is any non commutative polynomial in $p + 2q$ indeterminates,
- express the asymptotic statistics in elegant terms with

$$m = P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*).$$

Free Probability

Non commutative probability space

Definition of a *-probability space $(\mathcal{A}, \cdot^*, \tau)$

$\mathcal{A}$: unital $\mathbb{C}$-algebra,

$\cdot^*$: antilinear involution such that $(ab)^* = b^* a^*$ $\forall a, b \in \mathcal{A}$,

τ : linear form such that $\tau[\mathbf{1}] = 1$.

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Examples

- Commutative space: Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider $(L^\infty(\Omega, \mu), \bar{\cdot}, \mathbb{E})$
- Matrix spaces: $(\mathrm{M}_N(\mathbb{C}), \cdot^*, \frac{1}{N}\mathrm{Tr})$

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- τ is tracial: $\tau[ab] = \tau[ba]\ \forall a, b \in \mathcal{A}$,
- τ is a faithful state: $\tau[a^* a] \geq 0, \forall a \in \mathcal{A}$ and vanishes iff $a = 0$.
- $\mathcal{A}$ is a C^*-algebra: it is equipped with a norm $\| \cdot \|$ such that $\|a^* a\| = \|a\|^2 = \|a^*\|^2$.

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Examples

- Commutative space: Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider $(L^\infty(\Omega, \mu), \bar{\cdot}, \mathbb{E})$ and the infinity norm $\| \cdot \|_\infty$,
- Matrix spaces: $(\mathrm{M}_N(\mathbb{C}), \cdot^*, \frac{1}{N}\mathrm{Tr})$ with the operator norm $\|M\| = \sqrt{\rho(M^* M)}$.

Non commutative random variables

Proposition

If $a = a^*$ then there exists a compactly supported probability measure μ on $\mathbb{R}$ such that $\forall P$ polynomial $\tau\big[P(a)\ \big] = \int P d\mu$ and $\|a\| = \inf\Big\{A \geq 0 \ \Big|\ \mu(\ [-A, A]\ ) = 1\Big\}$.

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Definition

- Elements of $\mathcal{A}$: non commutative random variables (n.c.r.v.),
- Set of numbers $\tau\Big[P(\mathbf{a}, \mathbf{a}^*)\ \Big], \forall P$ non commutative polynomial : law of a family $\mathbf{a} = (a_1, \dots, a_p) \in \mathcal{A}^p$ (generalized moments).
- $\tau[P(\mathbf{a}_N, \mathbf{a}_N^*)] \underset{N \to \infty}{\longrightarrow} \tau[P(a, a^*)]\ \forall P$: convergence in law $\mathbf{a}_N \overset{\mathcal{L}^{n.c.}}{\longrightarrow} \mathbf{a}$.

The relation of freeness

Definition of freeness

The families of n.c.r.v. $\mathbf{a}_1, \dots, \mathbf{a}_p$ are free iff $\forall K \in \mathbb{N}$, $\forall P_1, \dots, P_K$ non commutative polynomials

$$\tau\Big[P_1(\mathbf{a}_{i_1}, \mathbf{a}_{i_1}^*) \dots P_K(\mathbf{a}_{i_K}, \mathbf{a}_{i_K}^*)\Big] = 0$$

as soon as $i_1 \neq i_2 \neq \dots \neq i_K$ and $\tau\big[P_k(\mathbf{a}_{i_k}, \mathbf{a}_{i_k}^*)\big] = 0$ for $k = 1, \dots, K$.

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Independence vs freeness

- if a and b are centered ($\tau[a] = \tau[b] = 0$) free n.c.r.v. then $\tau[abab] = 0$,
- if a and b are independent centered real random variables, $\mathbb{E}[abab] = \mathbb{E}[a^2]\mathbb{E}[b^2] = 0$ iff a or b are non random.

Voiculescu's asymptotic freeness

Consider

- $\mathbf{X}_N = (X_1^{(N)}, \dots, X_p^{(N)})$ be independent $N \times N$ GUE matrices
- $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_q^{(N)})$ $N \times N$ matrices independent with $\mathbf{X}_N$.

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Assumption

$\exists$ n.c.r.v. $\mathbf{y} = (y_1, \dots, y_q)$ s.t. for $\mathbf{Y}_N$ viewed as n.c.r.v. in $(\mathrm{M}_k(\mathbb{C}), .^*, \tau_N := \frac{1}{N}\mathrm{Tr})$ then when $N \to \infty$

$$\mathbf{Y}_N \xrightarrow{\mathcal{L}^{n.c.}} \mathbf{y} \quad \text{i.e. } \tau_N[P(\mathbf{Y}_N, \mathbf{Y}_N^*)] \to \tau[P(\mathbf{y}, \mathbf{y}^*)] \ \forall P.$$

Voiculescu's asymptotic freeness

Voiculescu (91)

Then

$\exists$ n.c.r.v. $\mathbf{x} = (x_1, \dots, x_p)$ such that

$$(\mathbf{X}_N, \mathbf{Y}_N) \xrightarrow{\mathcal{L}^{n.c.}} (\mathbf{x}, \mathbf{y}) \quad \text{i.e. } \tau_N[P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)] \to \tau[P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)] \ \forall P,$$

a.s. and in $\mathbb{E}$ when $N \to \infty$ and the law of $(\mathbf{x}, \mathbf{y})$ is given by

- $x_i = x_i^*$ and x_i has the semicircular law: $\tau[P(x_i)\,] = \int P d\sigma$
- the families $(x_1, \dots, x_p, \mathbf{y})$ are free.

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If $M_N = P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)$ is hermitian we obtain the convergence of its empirical spectral measure and the limit can be computed in term of $m = P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)$.

Strong asymptotic freeness

The problem

State assumptions on $\mathbf{Y}_N$ for which

$$\lim_{N\to\infty} \|P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\| = \|P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\|,$$

for all polynomial P.

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Previous results: for $\mathbf{Y}_N = \mathbf{0}$

- Haagerup and Thorbjørnsen (05): pioneering works,
- Schultz (0?): $X_N \sim$ GOE, GSE,
- Capitaine and Donati-Martin (0?): X_N Wigner with symmetric law of entries and a concentration assumption; X_N Wishart.

My result

Strong asymptotic freeness for $(\mathbf{X}_N, \mathbf{Y}_N)$

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2. Concentration assumption: $\exists \sigma > 0$ s.t. $\forall N$ the joint law of the entries of $\mathbf{Y}_N$ satisfies a Poincaré's inequality with constant σ/N i.e. $\forall f : \mathbb{R}^{2qN^2} \to \mathbb{C}$ of class C^1 s.t. $\mathbb{E}\big[\ |f(\mathbf{Y}_N)\ |^2\ \big] < \infty$ one has

$$\mathbb{V}\mathrm{ar}\big(f(\mathbf{Y}_N)\ \big) \leqslant \sigma/N\ \ \mathbb{E}\big[\ \|\nabla f(\mathbf{Y}_N)\ \|^2\ \big],$$

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3. Rate of convergence for generalized Stieltjes transforms.

Then $\lim_{N\to\infty} \|P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\| = \|P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\|$ for all polynomial P.

The linearization trick

To show $\forall P, \|P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\| \underset{N\to\infty}{\longrightarrow} \|P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\|$ a.s. , It is enough to show:
for any self adjoint degree one polynomial $L \in \mathrm{M}_k(\mathbb{C}) \otimes \mathbb{C}\langle \mathbf{x}, \mathbf{y}, \mathbf{y}^* \rangle$, for any $\varepsilon > 0$,

$$\mathrm{Sp}\big(\ L(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\ \big) \subset \mathrm{Sp}\big(\ L(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\ \big) + (-\varepsilon, \varepsilon)$$

almost surely for N large enough.

$$L(\mathbf{x}, \mathbf{y}, \mathbf{y}^*) = \sum_{i,j=1}^{k} \epsilon_{i,j} \otimes L_{i,j} = \begin{pmatrix} L_{1,1}(\mathbf{x}, \mathbf{y}, \mathbf{y}^*) & \dots & L_{1,k}(\mathbf{x}, \mathbf{y}, \mathbf{y}^*) \\ \vdots & & \vdots \\ L_{k,1}(\mathbf{x}, \mathbf{y}, \mathbf{y}^*) & \dots & L_{k,k}(\mathbf{x}, \mathbf{y}, \mathbf{y}^*) \end{pmatrix}.$$

Application

Convergence of spectra

Proposition

If $P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)$ is hermitian then $\forall \varepsilon, \exists N_0$ s.t. $\forall N \geq N_0$

$$\mathrm{Sp}\big(\ P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\ \big) \subset \mathrm{Sp}\big(\ P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\ \big) + (-\varepsilon, \varepsilon)$$

Diagonal matrices

Given $\mu_1, \dots, \mu_q$ compactly supported probability measures on $\mathbb{R}$ find $\mathbf{D}_N = (D_1^{(N)}, \dots, D_q^{(N)})$ for which the empirical spectral distribution of $D_j^{(N)}$ converges to μ_j, $j = 1, \dots, q$.

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- Cumulative distribution functions: $\forall t \in \mathbb{R}$, $F_j(t) = \mu_j(]-\infty, t])$, $j = 1, \dots, q$,
- Generalized inverses: $\forall u \in]0,1], F_j^{-1}(u) = \inf \big\{ t \in \ \big| \ F_j(t) \geq u \big\}$, $F_j^{-1}(0) = \lim_{u \to 0^+} F_j^{-1}(u)$.

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Define $\mathbf{D}_N = (D_1^{(N)}, \dots, D_q^{(N)})$ where for $j = 1, \dots, q$

$$D_j^{(N)} = \mathrm{diag} \left(F_j^{-1}\left(\frac{0}{N}\right), \dots, F_j^{-1}\left(\frac{N-1}{N}\right) \right).$$

Diagonal matrices

Proposition

If the support of the μ_j consists in a single interval then strong asymptotic freeness holds for $(\mathbf{X}_N, \mathbf{D}_N)$.

Wishart matrices

Definition

Wishart matrix with parameter r/s: $W_N = M_N M_N^*$ where $M_N = (M_{n,m})_{\substack{1\leqslant n\leqslant rN\\ 1\leqslant m\leqslant sN}}$, and

$$(\sqrt{2}\mathrm{Re}\ (M_{n,m}), \sqrt{2}\mathrm{Im}\ (M_{n,m})\ )_{1\leqslant n\leqslant rN, 1\leqslant m\leqslant sN}$$

is a centered Gaussian vector with covariance matrix $\frac{1}{rN}\mathbf{1}_{2rsN^2}$.

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Proposition

Strong asymptotic freeness holds for Wishart matrices with rational parameter (together with $\mathbf{Y}_N$) instead of GUE matrices

Non white Wishart matrices

Definition

Non white Wishart matrix: $Z_N = \Sigma_N^{1/2} W_N \Sigma_N^{1/2}$ where

- W_N Wishart,
- Σ_N non negative definite Hermitian.

Proposition

Strong asymptotic freeness holds for matrices $\mathbf{Z}_N = (Z_1^{(N)}, \dots, Z_p^{(N)})$ where the matrices $\Sigma_N^{1/2}$'s are of the diagonal form as before

Block matrices

Proposition

The operator norm of block matrices

$$\begin{pmatrix} P_{1,1}(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*) & \dots & P_{1,\ell}(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*) \\ \vdots & & \vdots \\ P_{\ell,1}(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*) & \dots & P_{\ell,\ell}(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*) \end{pmatrix},$$

converges a.s. as $N \longrightarrow \infty$.

Rectangular block matrices

"Channel matrix" in the context of telecommunication

$$H = \begin{pmatrix} A_1 & A_2 & \dots & A_L & \mathbf{0} & \dots & & \dots & \mathbf{0} \\ \mathbf{0} & A_1 & A_1 & \dots & A_L & \mathbf{0} & & & \vdots \\ \vdots & \mathbf{0} & A_1 & A_2 & \dots & A_L & \mathbf{0} & & \\ & & \ddots & \ddots & \ddots & & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & & \dots & \mathbf{0} & A_1 & A_2 & \dots & A_L \end{pmatrix},$$

$(A_l)_{1\leqslant \ell \leqslant L}$ are $n_R \times n_T$ matrices with i.i.d. complex Gaussian entries with mean m_ℓ and variance σ_ℓ^2/N.

Rectangular block matrices

Proposition

If $m_\ell = 0$, $\ell = 1..L$, then the norm of H converges for $n_R = rN$ and $n_T = tN$.

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If $m_\ell \neq 0$: finite rank deformation...

Thank you for your attention