Spectrum of large random Markov chains

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Large Random Matrices and their Applications
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- If A is normal ($AA^* = A^*A$) then $s_k(A) = |\lambda_k(A)|$ for all k
- Eigenvalues are very sensitive when A is not normal
- Singular values are more regular than eigenvalues

An elementary example (Śniady, Bai-Silverstein)

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- Pseudospectrum (see e.g. Trefethen & Embree 05)

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Theorem (Quartercircular law)

If X_{11} has finite positive variance σ^2 then a.s.

$$\nu_{\frac{1}{\sqrt{n}}\mathbf{X}} \xrightarrow[n\to\infty]{\mathscr{C}_b} \mathcal{Q}_\sigma$$

where $\mathcal{Q}_\sigma$ is the quartercircular law

$$x \mapsto \frac{1}{\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,\mathbf{1}_{[0,2\sigma]}(x).$$

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TECH: Truncation, centralization, Stieltjes recursion, fixed point

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TECH: Hermitization, log-potential, extremal singular values

Markov matrices

- Markov chain on $\{1, \ldots, n\}$

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}, \quad \mathbb{P}(X_{k+1} = j | X_k = i) = M_{ij}$$

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- $1 \in \mathrm{Eigenvalues}(M) \subset \{z \in \mathbb{C} : |z| \leqslant 1\}$

Random Markov matrices

We make **M** random:

$$M_{ij} = \frac{X_{ij}}{\rho_i} = \frac{X_{ij}}{X_{i1} + \cdots + X_{in}} \quad \text{where} \quad X_{ij} \geqslant 0 \text{ iid}$$

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How the spectrum of **M** behaves when n is large?

If X_{11} is exponential then

- **M** has iid rows following a Dirichlet law (uniform on Λ_n)
- **M** follows the uniform law on the Markov polytope Λ_n^n

Markov: singular values

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TECH: Product decomposition, Bai-Yin ULLN, Courant-Fischer, Quartercircular law theorem for **X**. Very similar to Aubrun 06

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Product decomposition into stabilized objects:

$$\mathbf{M} = \mathbf{D}\mathbf{X} \quad \text{where} \quad \mathbf{D} = \mathrm{Diag}(\rho_1^{-1}, \ldots, \rho_n^{-1})$$

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Bai-Yin ULLN when X_{11} has finite variance:

$$\max_{1\leqslant i\leqslant n}\left|\frac{\rho_i}{n} - \mathbb{E}(X_{11})\right| = o(1) \text{ a.s.}$$

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Courant-Fischer variational formulas:

$$s_n(\mathbf{D})s_k(\mathbf{X}) \leqslant s_k(\mathbf{M}) \leqslant s_1(\mathbf{D})s_k(\mathbf{X})$$

Markov: eigenvalues

Numerical experiment

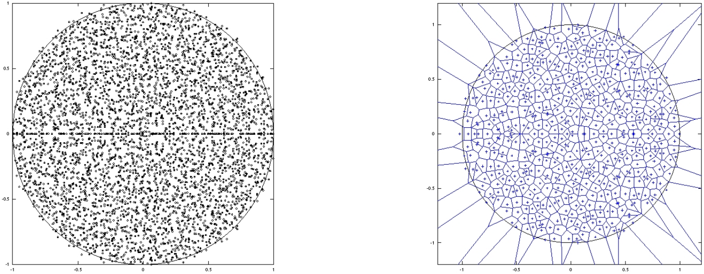

Figure: Eigenvalues of **M** when X_{11} Bernoulli $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $n = 250$

Markov: eigenvalues

Theorem (Circular law)

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TECH:

- Logartihmic potential and Hermitization
- Uniform integrability
  - Polynomial bounds on extremal singular values
  - Deviation of distances to subspaces (Talagrand)
  - Marchenko-Pastur type result for Hermitization

Logarithmic potential and Hermitization I

For a probability measure μ on $\mathbb{C}$ integrating $\log|\cdot|$ at ∞:

$$U_\mu : z \in \mathbb{C} \mapsto -\int_{\mathbb{C}} \log\left|z - z'\right| \mu(dz') \in (-\infty, +\infty]$$

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$$U_{\mu_{\mathbf{A}}}(z) = -\int_0^\infty \log(t)\,\nu_{\mathbf{A}-z\mathbf{I}}(dt)$$

Logarithmic potential and Hermitization II

Relation to Stieltjes transform:

$$S_\mu(z) = \int_{\mathbb{C}} \frac{1}{z' - z}\, \mu(dz') \overset{\mathcal{D}'}{=} (\partial_x - i\partial_y) U_\mu(z)$$

and thus

$$(\partial_x + i\partial_y) S_\mu \overset{\mathcal{D}'}{=} -2\pi\mu$$

Logarithmic potential and Hermitization II

Logarithmic energy (distribution of charges in $\mathbb{R}^2$):

$$\mathcal{E}(\mu) := \int_{\mathbb{C}} U_\mu(z)\,\mu(dz) = \iint_{\mathbb{C}^2} \log \frac{1}{|z-z'|}\,\mu(dz)\mu(dz')$$

Relation to circular law $\mathcal{C}_r$ (see e.g. book of Saff and Totik):

$$\mathcal{C}_r = \arg\min \left\{ \mathcal{E}(\mu) + \int_{\mathbb{C}} \frac{|z|^2}{r^2}\,\mu(dz) \right\}$$

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BIB: Logarithmic potentials with external fields : Saff-Totik 97
Voiculescu entropy (up to a sign) : Min energy - Max entropy
Brown spectral measure and Fuglede-Kadison determinant
Patlak-Keller-Segel PDE model

Logarithmic potential and Hermitization III

Large Deviation Principle when X_{11} is complex Gaussian:

$$\exp\left(\sum_{i\neq j}\log|\lambda_i-\lambda_j| - \sum_{i=1}^{n}|\lambda_i|^2\right)$$

$$\mathbb{P}(\mu_{\frac{1}{\sqrt{n}}\mathbf{X}}\in B)\approx \exp\left(-n^2\inf_{\mu\in B}\left\{\mathcal{E}(\mu)+\int_{\mathbb{C}}\frac{|z|^2}{\sigma^2}\,\mu(dz)\right\}\right)$$

BIB: Ben Arous-Guionnet 97, Ben Arous-Zeitouni 98

Logarithmic potential and Hermitization IV

Lemma (From Hermitization to spectral convergence)

Let $(\mathbf{A}_n)_{n\geqslant 1}$ *be a sequence of random matrices where* $\mathbf{A}_n$ *is* $n \times n$. *If for a.a.* $z \in \mathbb{C}$ *there exists a law* ν_z *on* $\mathbb{R}_+$ *such that a.s.*

- $(\nu_{\mathbf{A}_n - z\mathbf{I}})_{n\geqslant 1}$ *converges weakly to* ν_z
- $\log(\cdot)$ *is uniformly integrable for* $(\nu_{\mathbf{A}_n - z\mathbf{I}})_{n\geqslant 1}$

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then there exists a law μ *on* $\mathbb{C}$ *such that*

- *a.s.* $(\mu_{\mathbf{A}_n})_{n\geqslant 1}$ *converges weakly to* μ
- *for a.a.* $z \in \mathbb{C}$: $U_\mu(z) = -\int_0^\infty \log(t)\,\nu_z(dt)$

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- *$(\nu_{\mathbf{A}_n - z\mathbf{I}})_{n\geqslant 1}$ converges weakly to ν_z*
- *$\log(\cdot)$ is uniformly integrable for $(\nu_{\mathbf{A}_n - z\mathbf{I}})_{n\geqslant 1}$*

then there exists a law μ on $\mathbb{C}$ such that

- *a.s. $(\mu_{\mathbf{A}_n})_{n\geqslant 1}$ converges weakly to μ*
- *for a.a. $z \in \mathbb{C}$: $U_\mu(z) = -\int_0^\infty \log(t)\,\nu_z(dt)$*

TECH: Prohorov thm. + log-pot. unicity + Weyl inequalities:

$$\prod_{i=k}^{n} s_i(\mathbf{A}) \leqslant \prod_{i=k}^{n} |\lambda_i(\mathbf{A})| \quad \text{and} \quad \prod_{i=1}^{k} |\lambda_i(\mathbf{A})| \leqslant \prod_{i=1}^{k} s_i(\mathbf{A})$$

Logarithmic potential and Hermitization V

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  - Using LLN and polynomial upper bound on $s_1(\sqrt{n}\mathbf{M} - z\mathbf{I})$:

$$\int_1^\infty$$

  - Using polynomial lower bound on $s_n(\sqrt{n}\mathbf{M} - z\mathbf{I})$:

$$\int_0^{n^{-b}}$$

  - Using distance of rows to subspaces (via Talagrand ineq.)

$$\int_{n^{-b}}^1$$

Polynomial bound on least singular value

$$(\forall a, C > 0)(\exists b > 0)(\forall z, |z| \leqslant C)(\exists n_0)(\forall n \geqslant n_0)$$

$$\mathbb{P}(s_n(\sqrt{n}\mathbf{M} - z\mathbf{I}) \leqslant n^{-b}) \leqslant n^{-a}$$

In particular, for some $b > 0$ which may depend on C, a.s. for $n \gg 1$, $\sqrt{n}\mathbf{M} - z\mathbf{I}$ is invertible with $s_n(\sqrt{n}\mathbf{M} - z\mathbf{I}) \geqslant n^{-b}$.

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- We prove it when X_{11} has bounded density (drop that!)
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- Rigid structure of the model:

$$\mathbf{X} - \frac{z}{\sqrt{n}}\mathbf{D}^{-1} = \begin{pmatrix} R_1\mathbf{P}_1\mathbf{A}\mathbf{P}_1 \\ \vdots \\ R_n\mathbf{P}_n\mathbf{A}\mathbf{P}_n \end{pmatrix} \quad \text{where} \quad \mathbf{X} = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}$$

Polynomial bound on least singular value

$$\mathbf{A} = \mathbf{I} - \frac{z}{\sqrt{n}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$s_2(\mathbf{A}) = \cdots = s_{n-1}(\mathbf{A}) = 1 \quad \text{for } n \gg 1$$

and

$$\lim_{n\to\infty} s_n(\mathbf{A}) = \lim_{n\to\infty} s_1(\mathbf{A})^{-1} = \frac{\sqrt{2}}{\sqrt{2 + |z|^2 + |z|\sqrt{4 + |z|^2}}}.$$

Singular values of X: infinite variance

Behavior when $\mathbb{E}(X_{11}^2) = \infty$?

Singular values of X: infinite variance

Behavior when $\mathbb{E}(X_{11}^2) = \infty$?

No ULLN if $\mathbb{E}(X_{11}^2) = \infty$
No LLN if $\mathbb{E}(X_{11}) = \infty$

$$(\exists \alpha \in (0,2))(\forall t > 0) \quad \mathbb{P}(|X_{11}| > t) \approx t^{-\alpha}$$

New limiting distributions for **X**

Singular values of X: infinite variance

Theorem (Singular values)

For every $0 < \alpha < 2$ and $z \in \mathbb{C}$, there exists a law $\nu_{\alpha,z}$ on $[0,\infty)$ such that if $\mathbb{P}(|X_{11}| > t) \approx t^{-\alpha}$ then for some $a_n \sim n^{1/\alpha}$, a.s.

$$\nu_{a_n^{-1}X - zI} \xrightarrow[n\to\infty]{\mathscr{C}_b} \nu_{\alpha,z}$$

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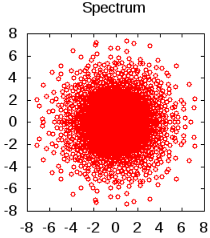
TECH: Aldous-Steele objective method, local weak convergence to bipartized Poisson Weighted Infinite Tree (lim. operator), spectrum via resolvent and Reed-Simon techniques.
BIB: for $z = 0$, alt. proof by Belinschi-Dembo-Guionnet 09

Eigenvalues of X: infinite variance

Numerical experiments



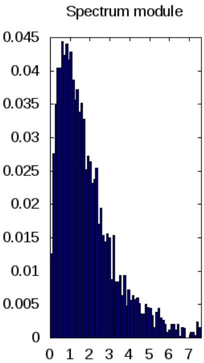


Figure: $n = 5000$, $\alpha = 1$, $X_{11} = \mathrm{Rademacher}(\pm 1) * (\mathrm{Unif}[0,1]^{-1/\alpha} - 1)$

Eigenvalues of X: infinite variance

Theorem (Eigenvalues)

For every $0 < \alpha < 2$, there exists a law μ_α on $\mathbb{C}$ such that if X is absolutely continuous with $\mathbb{P}(|X_{11}| > t) \approx t^{-\alpha}$ then for a deterministic sequence $a_n \sim n^{1/\alpha}$, a.s.

$$\mu_{a_n^{-1}\mathbf{X}} \xrightarrow[n\to\infty]{\mathscr{C}_b} \mu_\alpha$$

Moreover, μ_α is isotropic, absolutely continuous, with tail

$$c|z|^{2(\alpha-1)}e^{-\frac{\alpha}{2}|z|^\alpha} \quad \text{as} \quad |z| \to \infty.$$

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TECH: logarithmic potential, RDE via bipartization of PWIT

Additional results:

- Behavior of $s_1(\mathbf{M})$ and $\lambda_1(\mathbf{M})$ if $\mathbb{E}(X_{11}^4) < \infty$
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But still many open questions!